

# EXISTENCE OF TWO-PARAMETER CROSSINGS

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ABSTRACT. A Morse 2-function is a generic map from a manifold  $M$  of arbitrary finite dimension to a surface  $B$ . Its critical set is a smooth 1-submanifold of  $M$  that maps to a cusped, immersed collection of circles and (in the case that  $M$  and  $B$  are closed with nonempty boundary) properly embedded arcs in  $B$ . The aim of this paper is to explain exactly when it is possible to move these circles around in  $B$  and to give a few examples when  $M$  is a closed 4-manifold.

## 1. INTRODUCTION

The theory of singular fibrations of  $n$ -manifolds for  $n \geq 3$  has seen its share of applications in low-dimensional topology. For instance Morse functions, open books and Lefschetz fibrations all lead to depictions of 3- and 4-manifolds using surfaces (which are regular fibers) decorated with circles, and such discrete information has been a rich playground. Though its definitions go at least as far back as catastrophe theory, the Morse 2-function as a fibration structure is a promising invention of [ADK] that has seen some recent nontrivial development [AK, B1, B2, Be, BH, GK, H, L, W1, W2].

**Definition 1.1.** An *indefinite Morse 2-function* is a smooth map  $f$  from an  $n$ -manifold  $M$  to a 2-manifold  $B$  such that each critical point has the local model of an index  $k$  *indefinite fold* or an index  $k$  *indefinite cusp*, given by

$$\begin{aligned} (x_1, \dots, x_n) &\mapsto \left( x_1, -\sum_{i=2}^{k+1} x_i^2 + \sum_{i=k+2}^n x_i^2 \right) && \text{(fold)} \\ (x_1, \dots, x_n) &\mapsto \left( x_1, x_2^3 - 3x_1x_2 - \sum_{i=3}^{k+2} x_i^2 + \sum_{i=k+3}^n x_i^2 \right) && \text{(cusp)} \end{aligned}$$

for  $k \in \{1, \dots, n-2\}$ .

The local models for  $k=0$  and  $k=n-1$  (the *definite* fold and cusp) are omitted, though Theorem 2.4 could be easily adapted to handle those critical points as well. The reason for this is the definite locus can be eliminated from smooth maps when  $n=4$  by homotopy (see for example the earliest proof of this fact in [S]), and there are general existence results for indefinite one-parameter families of maps when  $n \geq 4$  [W2, GK]. To help understand folds and cusps, Figure 1 shows two disks in  $B$ , schematically depicting the fibration structure near a fold and a cusp for  $n=4$ . Bold arcs represent the image of the critical locus and a surface is pictured in the region of regular values that have that surface as their preimage. Tracing point preimages above a horizontal arc from left to right gives the foliation of  $\mathbb{R}^3$  by hyperboloids, with a double cone above the fold point. The circle (drawn on the cylinder to the left) that shrinks to the cone point is called the *vanishing cycle* for that critical arc. Each cusp point is a common endpoint of two open arcs of fold

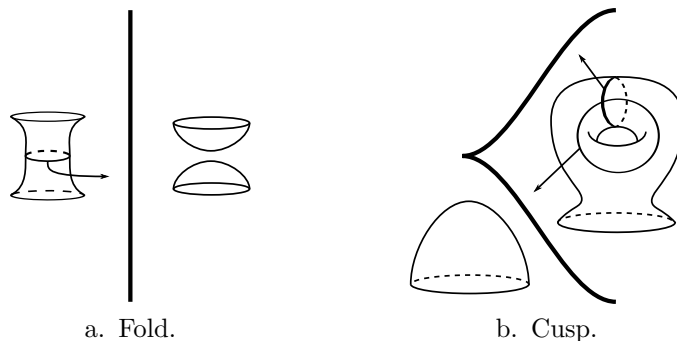


FIGURE 1. Critical points of purely wrinkled fibrations.

points for which the two vanishing cycles must transversely intersect at a unique point in the fiber. See [GK] for a more detailed background on these maps.

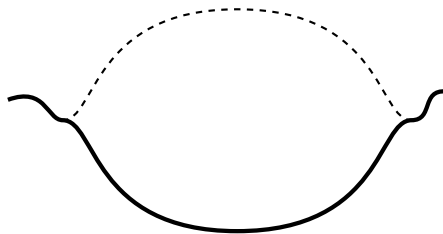


FIGURE 2. A fold arc (in bold) runs along the boundary of a disk  $\Delta$  in the target space of  $f_0$  in preparation to move across  $\Delta$  by some deformation of  $f_0$ , eventually to coincide with the dotted line (other critical points omitted).

The main point of this paper is the complete characterization of when a proposed movement in the critical image of a wrinkled fibration, similar to a sequence of Reidemeister moves for knot projections, can be achieved by a deformation. Such homotopies are generically sequences of *2-parameter crossings*, a term that appeared in [GK, Section 2]. In their paper and in this paper, it replaces the less accurate term *isotopy* that appeared in [L] and [W1, Section 2.4.1], which in those papers was a sequence of 2-parameter crossings. To describe the general situation, suppose there is a purely wrinkled fibration  $f_0: M^4 \rightarrow D^2$  and an embedded disk

$$\Delta: \{z \in \mathbb{C} : |z| \leq 1\} \rightarrow D^2$$

smoothly embedded except at the two corners on either side, such that, for some fold arc  $\varphi_0 \subset M_0$ , the restriction

$$f_0|_{\varphi_0}: \varphi_0 \rightarrow \Delta(\{|z| = 1, \operatorname{Im} z \leq 0\})$$

is a diffeomorphism (see Figure 2; sometimes  $\Delta$  also denotes its image). The problem is to determine when there is a deformation  $f_t: M_I \rightarrow D_I^2$  such that the image of each point  $\Delta(z) \in f_0(\varphi_0)$  in this fold arc follows the path  $\Delta(t\bar{z} + (1-t)z)$  while the rest of the critical image remains fixed. Theorem 2.4 gives a necessary and

sufficient criterion for this to occur, and the following propositions present a few standard cases in which a proposed 2-parameter crossing always exists. Many of those propositions may generalize to  $n > 4$ ; the restriction to  $n = 4$  is due to the nonzero number of applications anticipated by the author.

## 2. MAIN THEOREM AND APPLICATIONS TO 2-PARAMETER CROSSINGS

Consider a Morse function  $f: M^n \rightarrow [0, 4]$  with two isolated critical points  $p_1$  and  $p_2$ , with  $f(p_i) = i$ , and equip  $M$  with a Riemannian metric. If the ascending manifold  $Z$  of  $p_1$  is disjoint from the descending manifold  $Y$  of  $p_2$  (where critical points are considered part of their ascending and descending manifolds), then there is a neighborhood  $\nu Z$  on which the restriction of  $f$  has exactly one critical point  $p_1$ , and a homotopy of  $f$  supported on  $\nu Z$  that sends  $f(p_1)$  to 3. In Figure 2, there is a one-parameter family of smooth functions, finitely many of which may not be Morse, and a one-parameter family  $Z$  of ascending manifolds for the one-parameter family of Morse critical points given by  $\varphi_0$ . If  $Z$  is disjoint from the one-parameter family of descending manifolds coming from all other critical points in  $\Delta$ , then the proposed homotopy exists over every vertical arc  $\{z = \text{const}\}$ , allowing the fold arc to pass over  $\Delta$ . Here follows a more precise statement using the language of [BH, Section 2.3].

**Definition 2.1** (Disjointness condition). Let  $\mathcal{H}$  denote a horizontal distribution for the Morse 2-function  $f_0$  over a neighborhood of  $\Delta$ , obtained by taking the orthogonal complement  $\ker(df_0)^\perp$  with respect to some Riemannian metric on  $f_0^{-1}(\Delta)$ . Consider the union over all regular values  $p$  of  $f_0$ ,

$$V = \bigcup_p V_0^{\mathcal{H}}(\gamma_p^{up}) \cup V_0^{\mathcal{H}}(\gamma_p^{down}),$$

where the smooth embeddings  $\gamma_p^{up}, \gamma_p^{down}: [0, 1] \rightarrow \Delta$  travel from  $p$  vertically up or down, respectively, to the two boundary arcs of  $\Delta$ . Let  $Z$  be the points in  $V$  that run into  $\varphi_0$  and let  $Y$  be the points in  $V$  that run into  $\text{crit}(f_0) \setminus \varphi_0$ . The disjointness condition is that there exists an  $\mathcal{H}$  such that  $Y$  and  $Z$  are bounded away from each other in the preimage of  $\Delta$ .

In this definition, it is acceptable for the terminal point of  $\gamma_p^{down}$  to be a singular fiber over  $f(\varphi_0)$  because we only consider the vanishing set  $V_0$  of points in the regular fiber  $f_0^{-1}(p)$  that run into critical points (not the ones that emerge from critical points). Similarly, there may be arcs  $\gamma_p^{up}$  whose endpoint is a critical value. Vanishing sets appear in the definition instead of ascending and descending manifolds because they are defined. For example, suppose  $\gamma_p^{up}$  is tangent to the image of a fold arc. In that case, the restriction of  $f_0$  to the preimage of  $\gamma_p^{up}$  is not a Morse function, so it is not appropriate to speak of the descending manifold of the critical point of  $f_0$  above that tangency. On the other hand, the vanishing set is defined in [BH] and has the same properties related to pushing around critical values.

**Remark 2.2.** A disk  $\Delta$  and a horizontal distribution  $\mathcal{H}$  satisfying the disjointness condition over  $\Delta$  can be used to specify a deformation suggested by  $\Delta$ . Let  $N$  be a neighborhood of the vanishing set of  $\varphi_0$  over a neighborhood  $\nu\Delta$  of  $\Delta$  (note  $N$  is a ball, as discussed in [W1, Section 2.3.1]). Choose local coordinates near  $N$  so that  $f_0|_N$  is given by the local model for folds. Then  $x_1$  parameterizes  $\varphi_0$ , and the level sets for suitably parameterized vertical arcs foliating  $\nu\Delta$  are the level sets

of  $-\sum_{i=2}^{k+1} x_i^2 + \sum_{i=k+2}^n x_i^2$ . Such local coordinates exist because  $N$  consists of regular points, with the exception of  $\varphi_0$ , by the disjointness condition. Now, restricting to  $N$ , Figure 2 is merely the image of the local model for folds, and there is certainly a deformation of the local model  $N \rightarrow \nu\Delta$ , fixing the map near  $\partial\overline{N}$ , realizing the suggested movement in  $S^2$ . Call it  $f_t^N$ ,  $t \in [0, 1]$ . Then the deformation is given by

$$f_t(x) = \begin{cases} f_t^N(x) & x \in N \\ f_0(x) & x \notin N. \end{cases}$$

**Remark 2.3.** One further observation is that  $Z \cap Y$  always maps to a family of vertical arcs connecting the two sides of  $\Delta$ , because the union of paths traced by any point in  $f_0^{-1}(p)$  above  $\gamma_p^{up}$  and its reverse coincides with that of  $\gamma_p^{down}$ , and its reverse.

**Theorem 2.4.** The disjointness condition of Definition 2.1 is equivalent to the existence of a deformation as suggested by Figure 2.

*Proof.* The disjointness condition is merely a restatement of the requirement that the relevant critical sets of the restrictions of  $f_0$  to the vertical arcs in  $\Delta$  simultaneously satisfy the same disjointness condition for switching critical points of smooth real-valued functions.  $\square$

Here follow some examples of how to use Theorem 2.4 to verify a proposed movement of critical arcs in the base when  $n = 4$ . Note that indefinite folds in this case have a higher-genus and lower-genus side because the regular fibers of such maps are closed, orientable surfaces. Existence results for various types of what are called *Reidemeister-2 fold crossings* in [GK] and  $R_2$  deformations in [W2] are collected into the following proposition.

**Proposition 2.5** (Existence of Reidemeister-2 fold crossings).

- (1) There exists a deformation as suggested in Figure 3 if one of the following two conditions occur.
  - (a) The point  $p$  is on the lower-genus side of at least one of the pictured fold arcs.
  - (b) The point  $p$  is on the higher-genus side of both fold arcs, and there exist embedded reference paths from  $p$  to each fold arc within  $\Delta$  for which the measured vanishing cycles in  $f_0^{-1}(p)$  are disjoint.
- (2) In Figure 4, suppose the fiber genus of each fiber component above  $p$  is at least 2, and that  $p$  lies on the higher genus side of exactly one of the fold arcs, say  $\varphi_0$ . Then the suggested  $R_2$  deformation exists if and only if the vanishing cycles at  $\Delta(\pm 1)$  match, as measured with reference paths from  $\Delta(i)$ .
- (3) There always exists an  $R_2$  deformation as suggested by Figure 4 when  $p$  is on the lower-genus side of both fold arcs, even when there are cusps in the boundary of the lune.
- (4) In Figure 4, suppose  $p$  lies on the higher-genus side of both fold arcs, all regular fibers are connected, and the fiber above  $p$  has genus at least 2. Then there exists an  $R_2$  deformation as suggested by the figure.

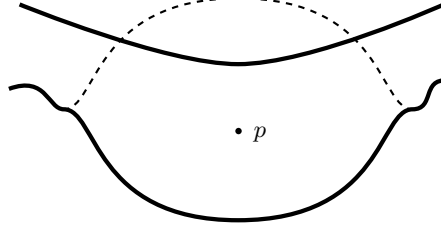


FIGURE 3. Preparing to push a bit of a fold arc past another, in what is commonly called a *finger move*, for Proposition 2.5(1).

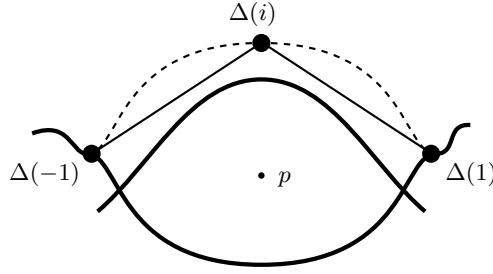


FIGURE 4. The disk  $\Delta$  with lower fold arc  $f_0(\varphi_0)$  in preparation for an  $R_2$  deformation in items (2)-(4) of Proposition 2.5.

*Proof.* The typical approach for a statement like Assertion (1) will be to show that there is a choice of horizontal distribution such that  $Z \cap Y = \emptyset$ . First, note that  $p$  must either lie on the lower-genus side of at least one fold arc which we will call  $A$  (case (a)), or on the higher-genus side of both (case (b)). In either case, the deformation exists if and only if there is a vertical arc  $v \subset \Delta$  connecting the two fold arcs that is disjoint from  $f_0(Z \cap Y)$ , because in that case one could shrink  $\Delta$  so that its interior is a tubular neighborhood of  $v$  whose preimage satisfies the disjointness condition.

For case (1a),  $Z \cap Y$  is 1-dimensional if  $p$  lies on the lower-genus side of exactly one of the fold arcs, and by Remark 2.3 it maps to a disjoint union of vertical arcs in  $\Delta$ . If  $p$  lies on the lower-genus side of both of the fold arcs then  $Z \cap Y$  is 0-dimensional, the intersection of two surfaces in  $M_I$ . However, Remark 2.3 implies the dimension of  $Z \cap Y$  is at least 1, for any  $\mathcal{H}$ . For this reason,  $Z \cap Y$  must be empty in this case.

**Remark 2.6.** If the previous paragraph sounds strange, here is a digression, giving another way to explain its conclusion: Any arc in  $Z \cap Y$  is simultaneously the cocore of a three-dimensional 2-handle and the core of a 3-dimensional 1-handle, which is not only non-generic in the space of Morse functions because it is not Morse-Smale, but is also non-generic in the space of one-parameter families of Morse functions (parameterized by the horizontal direction  $s$  in Figure 3): Over any horizontal line  $\ell(s)$ , the pairs of points that run into the top and bottom fold arcs trace paths in the 3-manifold (fiber)  $\times \ell$ , and an arc in  $Z \cap Y$  is an arc of intersections between these paths. Given such an arc  $\gamma \subset Z \cap Y$ , one merely perturbs  $\mathcal{H}$  so that the  $s$ -value at which one pair of paths crosses  $\gamma$  is different than the  $s$ -value for the other, and

this can be achieved all along  $\gamma$  by performing essentially the same perturbation crossed with  $\gamma$ . This concludes the digression.

In either case, the dimension of  $Z \cap Y$  is no greater than 1, so there is at least one vertical arc in  $\Delta$  whose preimage in  $Z$  is disjoint from  $Y$ , and so this arc has a neighborhood whose preimage in  $Z$  is disjoint from  $Y$ , and we perform the finger move in this neighborhood.

For case (1b), the assumption is that  $Z \cap Y = \emptyset$  in the fiber above  $p$ . In the language handlebodies, we have a three-dimensional 1-handle attachment followed by a 2-handle attached along a circle that can up to isotopy is disjoint from the belt circle of the 1-handle. In this case, it is also known that the order of handle attachment can be switched by a homotopy of the corresponding Morse function, then un-switched, which similarly gives the suggested  $R_2$  deformation.

For the  $(\Leftarrow)$  part of Assertion (2), assume the fiber above  $p$  is connected, since the result obviously holds if the vanishing cycles of the two fold arcs, as measured from  $\Delta(i)$ , lie in different fiber components. Denote by  $F$  the generic fiber over  $\Delta(i)$  and  $F'$  the fiber over  $p$ . Also, denote by  $q, q'$  the pair of points in  $F'$  corresponding to the fold arc not containing  $\varphi_0$ , again assumed to be stationary. Because of the matching vanishing cycles, it makes sense to talk about a single vanishing cycle  $v$  of the fold arc containing  $\varphi_0$  that lives in  $F$ , measured near either end of  $\varphi$  (and also in  $F'$  for those points actually in  $\varphi$ ). Choosing small  $\epsilon > 0$ , the representatives of  $v$  as measured at  $\Delta(\pm 1 \mp \epsilon)$  are isotopic in  $F' \setminus \{q, q'\}$  because they are isotopic in  $F$ . The surface  $F' \setminus \{q, q'\}$  has free fundamental group, in which elements whose representatives differ by crossing  $q, q'$  are distinct. This implies that each crossing of  $v$  over  $q$  or  $q'$  (that is, each point in  $Z \cap Y$ , which is generically a 1-submanifold of  $M$ ) is accompanied by a canceling crossing (paired, say, by letting the endpoints of reference paths travel along horizontal lines across the central lune to get one-parameter families of points and vanishing cycles in  $F'$ ). This allows the disjointness condition to be satisfied by a homotopy of  $\mathcal{H}$  that cancels the crossings by identifying vertical arcs in  $\Delta$  in pairs. The  $(\Rightarrow)$  part of Assertion (2) is clear; see Example 2.8 for a few examples.

For Assertion (3), after moving all cusps into the higher-genus sides of the two folds, the proof of the analogous case of Assertion (1a) (the last two sentences above Remark 2.6) goes through word-for-word.

Now for Assertion (4). If  $Z$  and  $Y$  have no intersections above the lune  $B \subset \Delta$  bounded by the fold arcs, then it is possible to choose  $\Delta$  small enough to make  $Z \cap Y = \emptyset$ . Assume without loss of generality that  $Z$  and  $Y$  intersect transversely, and let  $p \in L$ . Since all fibers are connected, the two vanishing cycles as measured at  $p$  using the reference paths  $\gamma_p^{up}, \gamma_p^{down}$  are homologically distinct and disjoint for all  $p$  sufficiently close to the fold crossings. In this situation, any point of intersection between  $Z$  and  $Y$  is reflected by an intersection between the vanishing cycles in the fiber using vertical reference paths, and this point is one of a pair that forms the corners of a family of lunes, one in each fiber, each of whose boundary is a pair of arcs, one in each of the two vanishing cycles. By a homotopy of  $\mathcal{H}$ , these lunes can be simultaneously contracted, eliminating the intersections of  $Z$  and  $Y$  from  $L$ . Now the proposition follows from Theorem 2.4.  $\square$

Here is one more existence result for finger moves involving cusps.

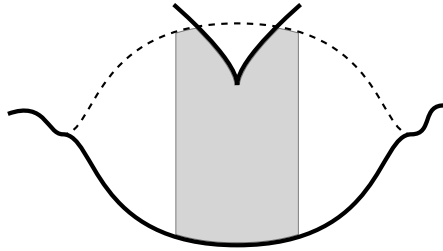


FIGURE 5. Pushing a fold across a cusp. Here,  $\Delta$  could lie on the higher or lower genus side of  $f_0(\varphi_0)$ .

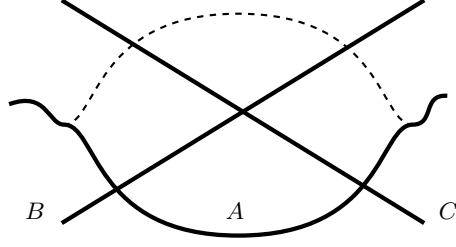
**Proposition 2.7.** There always exists a *cusp-fold crossing* deformation as suggested by Figure 5.

*Proof.* The proof of this result is just the same as that of Proposition 2.5(1a), except that  $Z$ , instead of being a pair of embedded disks given by the fiberwise feet of a 3-dimensional 1-handle, is an immersed pair of disks whose immersion locus is an arc of double points limiting to the cusp, and the argument applies after perturbing  $\mathcal{H}$  to eliminate intersections between  $Z$  and  $Y$  along this arc similarly to Remark 2.6.  $\square$

**Example 2.8.** Here are a few examples illustrating subtleties surrounding Proposition 2.5(2). Suppose that above  $\Delta^{-1}(\{\text{Re } z = 0\})$  there is an arc of intersection between  $Z$  and  $Y$ , where the vanishing cycle of  $\varphi_0$  crosses once over one of the points that runs into the other fold arc, traveling along the reference path  $\{\text{Re } z = 0\}$ . Then up to isotopy the two vanishing cycles mentioned in Proposition 2.5(2) differ by sliding over a disk bounded by the other fold arc's vanishing cycle, so that the  $R_2$  deformation would result in the particular impossibility that validates the  $(\Rightarrow)$  part of the statement:  $\varphi_0$  would be free of intersection points with other folds and have vanishing cycles at either end that are not isotopic in the fiber above  $\Delta(i)$ .

Another more subtle problem occurs when the regular fiber over the lune is a torus or a sphere. In this case, there can be many vertical arcs in  $f_0(Z \cap Y) \cap \Delta$  in which the vanishing cycle crosses one of the points with the same orientation (think of a meridian of a torus traveling around and around the torus, repeatedly crossing some marked point). Unlike the previous example, the vanishing cycles of  $\varphi_0$  at  $\Delta(\pm 1)$  can match, and the base diagram one might expect from such a move might not have obvious inconsistencies, but the proposed movement of fold arcs in the base is still not valid by Theorem 2.4.

In all these cases,  $Z \cap Y$  is generically 1-dimensional: over every vertical arc (oriented downward) in  $\Delta$  the Morse function corresponds to a pair of consecutive 3-dimensional 2-handle attachments, and every point of  $Z \cap Y$  is an intersection between the cocore of the first 2-handle (1-dimensional in each arc, sweeping out a 2-dimensional subset of  $M_0$ ) and the core of the second (2-dimensional in each arc, sweeping out a 3-dimensional subset of  $M_0$ ). Certainly there is no way to switch the order of handle attachments while keeping the intersection between core and cocore inside the lune, and (as the first example shows) there is no general way to move a vertical arc of intersections out of  $\Delta$  by a homotopy.

FIGURE 6. An  $R_3$  deformation.

**Proposition 2.9** (Existence of  $R_3$  deformations). There exists an  $R_3$  deformation when there are no cusps in the triangle and the minimal fiber genus in a neighborhood of the triangle is at least 2.

*Proof.* Figure 6 depicts the closure  $\Delta$  of a neighborhood of a triangle of fold arcs. Foliate  $\Delta$  by a family of vertical paths, each of which can be considered the target of a member of a one-parameter family of Morse functions. There are four cases, according to the number  $n$  of fold arcs that have the central triangle on their lower-genus sides. In all cases, the restriction of the map to the preimage of each leaf is a Morse function with three critical points and two or three critical values. The depicted  $R_3$  deformation can be thought of as affecting the vertical order of the critical values of these Morse functions in various ways. For any  $R_3$  deformation, we let the disk  $\Delta$  be the closure of a regular neighborhood of the triangle formed by the three fold arcs, with  $f_0(\varphi_0)$  coinciding with  $A$ . Since the fiber is connected with genus at least 2 over every regular value, the vanishing cycles of any one of the three folds as measured by different reference paths from the same arbitrarily chosen point can be assumed to be equal by appropriate choice of horizontal distribution, not just isotopic. Example 2.10 shows why this is a necessary assumption.

The case  $n = 3$  can be satisfied as in Proposition 2.5(3): though they would be intersections between two surfaces in a 4-manifold, intersections between the ascending manifold of  $A$  and the descending manifold of  $B$  or  $C$  are not generic.

For  $n = 2$ , without loss of generality suppose the triangle is on the higher-genus side of  $B$  and the lower genus sides of  $A$  and  $C$ . As in the paragraph above Proposition 2.9, generically the intersection between the ascending manifold of  $A$  and the descending manifold of  $B$  maps to a collection of vertical line segments that are not clearly removable. However, it is possible to move such a line segment to the left across  $C$ , out of the triangle, to lie in the higher-genus side of  $C$ . Intersections coming from  $A$  and  $C$  are ruled out as in the case  $n = 3$ .

For  $n = 1$ , without loss of generality suppose the triangle is on the higher-genus sides of  $B$  and  $C$  and the on the lower genus side of  $A$ . As in the case  $n = 2$ ,  $Z \cap Y$  is a finite number of line segments connecting  $A$  to  $B$  (or  $A$  to  $C$ ). Since  $B$  and  $C$  intersect, their vanishing cycles  $b_p, c_p$  (as measured from any reference point  $p$  in their higher-genus sides) are isotopic to disjoint circles. For this reason, for every  $p$ , every transverse intersection  $b_p \cap c_p$  is one of a pair of corners of a lune in  $F_p$ , with one side lying in  $b_p$  and the other side lying in  $c_p$ . By smoothness of the fibration map, these lunes can be chosen to vary smoothly with  $p$ . For all  $p$  on the higher genus side of both  $B$  and  $C$ , simultaneously collapse the lunes by an isotopy sending the  $b_p$  side to the  $c_p$  side of each. What results in each fiber is a



collection of arcs (perhaps even all of  $B$  and  $C$  if they are isotopic) on which the intersections of the fiber with  $B$  and  $C$  coincide. These arcs vary smoothly with  $p$  and it is possible to perturb  $b_p$  smoothly over all  $p$  to replace each arc with two parallel arcs, one for  $b_p$  and one for  $c_p$ . Now points in  $Z \cap Y$  coming from  $A$  and  $B$  are necessarily disjoint from  $c_p$  for all  $p$  in the triangle, so they can be pushed out of the triangle across  $C$ , and those points in  $Z \cap Y$  coming from  $A$  and  $C$  can be pushed out across  $B$ .

The  $n = 0$  case, which is not used in this paper because it is strictly genus-decreasing, is included for completeness. Here,  $Z \cap Y$  is generically a surface in  $M$  mapping to a codimension 0 subset of  $\Delta$ , naturally interpreted as a collection of intersection points between vanishing cycles. Let  $a_p, b_p, c_p$  be the three vanishing cycles for  $A, B, C$ , respectively, measured from a reference fiber  $F_p$  over a point  $p$  in the triangle. All three fold arcs intersect pairwise, so that  $a_p, b_p, c_p$  are isotopic to pairwise disjoint circles in  $F$ . As in the  $n = 1$  case, it is possible to arrange for  $b_p$  and  $c_p$  to be disjoint in every fiber on the higher-genus sides of  $B$  and  $C$ . In particular, this eliminates any triple intersections in  $Z \cap Y$ , so that points in that set coming from  $A$  and  $C$  are disjoint from  $B$ , so they can be pushed across  $B$ , and similarly those coming from  $A$  and  $B$  can be pushed across  $C$ . The three remaining bits at the angles opposite the corners of the triangle can be assumed free of intersections by choosing  $\Delta$  small enough.  $\square$

**Example 2.10.** This example, adapted from one due to the anonymous referee for [W2], shows why the genus assumption in Proposition 2.9 is necessary. Suppose  $n = 0$ , so that the triangle is on the higher-genus side of all three fold arcs, the regular fiber over any point inside the triangle has genus 1, and that the vanishing cycles for all three fold arcs are isotopic in the central fiber to a meridian. Choose the reference fiber that lies over the point  $p$  at the center of the triangle, which we suppose is equilateral. For a short interval of radial reference paths from  $p$  to points in the interior of one of the sides of the triangle, suppose the vanishing cycle travels once around the torus. In this situation, it seems the disjointness condition between  $Z$  and  $Y$  cannot be arranged by pushing intersections out of the triangle.

**Proposition 2.11** (Genus-increasing movements of an isolated fold arc). There exists a sequence of two-parameter crossings as suggested by Figure 2 when the lune is on the lower-genus side of  $f(\varphi_0)$  and  $\iota \cap \varphi_0 = \emptyset$  and all components of all regular fibers inside the lune have genus at least 2.

*Proof.* The proposed movement exists by the genus-increasing 2-parameter crossing propositions, namely Proposition 2.5(1a,2,3) for  $R_2$  deformations, Proposition 2.7 for cusp-fold crossings, and  $n \neq 0$  cases of Proposition 2.9 for  $R_3$  deformations. The hypotheses of these results are satisfied automatically, except those of Proposition 2.9 (simply observe that any cusps in the triangle can be first pushed into the higher-genus side of  $\varphi_0$ , which by hypothesis has no cusps) and Proposition 2.5(2).

In the context of Proposition 2.5(2) and in Figure 4, the fold arc  $\varphi_0$  for the present proposition (call it  $\varphi^{\text{present}}$ ) would be the upper fold arc, while the  $\varphi_0$  mentioned in Proposition 2.5(2) (call it  $\varphi^{(2)}$ ) would be the lower fold arc. One may choose the movement of  $\varphi^{\text{present}}$  as beginning with a pair of finger moves that exist by Proposition 2.5(1a), so that the first pair of intersections as in Proposition 2.5(2) to form during the movement comes from the two finger moves. By appropriately choosing a path in  $M_0$  for the tip of the second finger move to trace, one may

choose the location of the two points that become identified in the fiber when  $\varphi^{\text{present}}$  passes by; in particular, near the crossings of Figure 4. In this way, it is possible to arrange for the two vanishing cycles of  $\varphi^{(2)}$  to match, allowing an application of Proposition 2.5(2) to unify the two fingers into one thick finger. The movement continues with another finger move, chosen appropriately as before, and so on until the proposed movement is finished.  $\square$

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